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Critical thresholds in the semiclassical limit of 2-D rotational Schrödinger equations

Hailiang Liu

Abstract. We consider a two-dimensional convection model augmented with the rotational Coriolis forcing, centrifugal forcing as well as the quadratic potential $V(x)$, $\partial_t U + (U - \Omega x^\perp) \cdot \nabla_x U = -\Omega U^\perp - \nabla_x V$, with a fixed $\Omega > 0$ being the rotational frequency. This model arises in the semiclassical limit of the *Gross–Pitaevskii* equation for Bose–Einstein condensates in a rotational frame. We investigate whether the action of dispersive rotational forcing complemented with the underlying potential prevents the generic finite time breakdown of the free nonlinear convection. We show that the rotating equations admit global smooth solutions for and only for a subset of generic initial configurations. Thus, the global regularity depends on whether the initial configuration crosses an intrinsic critical threshold, which is quantified in terms of the initial spectral gap associated with the 2×2 initial velocity gradient, $\lambda_2(0) - \lambda_1(0)$, $\lambda_j(0) = \lambda_j(\nabla_x U_0)$ as well as the initial divergence, $\mathrm{div}_x(U_0)$.

We also prove that for the case of isotropic trapping potential the smooth velocity field is periodic if and only if the ratio of the rotational frequency and the potential frequency is a rational number. The critical thresholds are also established for the case of repulsive potential. Finally the position density and the velocity field are explicitly recorded along the deformed flow map.

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1. Introduction and statement of main results

In this paper, we study the regularity of a 2-D convection model augmented by a rotational force as well as the underlying potential $V = V(x)$,

$$\partial_t U + (U - \Omega Jx) \cdot \nabla_x U = -\Omega JU - \nabla_x V, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.1}$$

subject to the smooth initial velocity, $U(0, x) = U_0(x)$. The associated position density ρ is governed by a forced transport equation

$$\partial_t \rho + \nabla_x \cdot (\rho U) = \Omega \langle Jx, \nabla_x \rho \rangle. \tag{1.2}$$

Here Ω is a given positive constant which signifies the frequency of the rotational frame. The potential V is a given real-valued function. The unknowns are the local density $\rho = \rho(x,t)$ and the velocity field $U = U(x,t)$. The system (1.1)–(1.2) describes the dynamic behavior of some prototypical rotational physical flows. Particularly this model also arises in the semiclassical limit of the rotational Schrödinger equation for Bose–Einstein condensates(BEC). Historically since the first experimental achievement of Bose–Einstein condensates in atomic gases in 1995, many properties of BECs have been studied experimentally and theoretically. The properties of a BEC at temperature very much smaller than the critical condensation temperature T_c are usually modeled by a Schrödinger equation for the macroscopic wave function known as the Gross–Pitaevskii equation (GPE). The vortices are produced when putting the BEC in a rotational frame, and the trapping potential can therefore be made time-independent, see e.g. [1, 4] and references therein.

The re-scaled GPE with ignored atomic interaction reads

$$i\epsilon\partial_t\psi(x,t) = -\frac{\epsilon^2}{2}\Delta\psi(x,t) + i\epsilon\Omega\langle x^\perp, \nabla_x\psi\rangle + V(x)\psi(x,t), \tag{1.3}$$

where ψ is the complex wave field, $V(x) = \frac{1}{2}|\omega\cdot x|^2$ is the trapping potential, and $\epsilon > 0$ denotes a re-scaled Planck constant. Here the regime of interest is the so called *semiclassical regime* as ϵ tends to zero. Customarily the high frequency field is sought in the WKB form

$$\psi = A(x,t)\exp\left(iS(x,t)/\epsilon\right).$$

Insertion of this ansatz into the wave equation (1.3) yields the leading order approximation (with $O(\epsilon^2)$ terms ignored) for the phase function $S = S(x,t)$,

$$\partial_t S + \frac{1}{2}|\nabla_x S|^2 - \Omega\langle Jx, \nabla_x S\rangle + V(x) = 0,$$

which upon differentiation gives the equation (1.1) for phase gradient $U = \nabla S$, with the position density $\rho := A^2$ satisfying the transport equation (1.2).

Apart from the Bose–Einstein condensate, problem (1.3) with highly oscillatory initial data arises in many contexts in classical wave propagation, such as the paraxial approximation of forward propagating waves [11], radio engineering [12], laser optics [24], underwater acoustics [25], the investigation of light and sound propagating in turbulent atmosphere [26], seismic wave propagation in the earth's crust [22], etc. In these applications, the potential V is explicitly related to the feature of the propagating medium.

Note that in equation (1.1) there is a competition between the finite-time breakdown dynamics driven by the nonlinear convection and the balancing act of rotational forcing as well as the underlying potential. As remarked in [18], in the classical hydrodynamic system three prototypes of forcing mechanisms which often come into effect are dissipation, relaxation and dispersion. It is well known that a sufficiently large amount of either dissipation or relaxation is crucial for a global smooth solution for a rich enough class of initial data. In both cases of dissipation and relaxation, global existence is secured by enforcing a sufficiently large amount of energy decay. Dispersive forcing, however, is different. We point out that when singularity arises at the hydrodynamic level the physically relevant solution in this context is usually multi-valued, consult e.g., [7, 13, 28] for a recent development of numerical methods for computing multi-valued physical observables in the semiclassical limit of Schrödinger equations.

The system (1.1)–(1.2) admits a global energy invariant in time, which depends on the amplitude of rotation encoded by the rotational frequency Ω as well as the potential V in (1.1). A formal calculation shows that the global invariance of the generalized energy follows

$$E(t) := \int_x \left[\frac{1}{2}\rho(t,x)|U(t,x) - \Omega Jx|^2 + \rho(V(x) - \frac{\Omega^2}{2}|x|^2)\right] dx = E(0).$$

However, this global invariant alone is not enough for justifying the time-regularity of the velocity field.

Note that if the potential and the centrifugal force can be ignored, system (1.1) will reduce to

$$\partial_t U + U \cdot \nabla_x U = -\Omega J U,$$

for which regularity effect of the rotational forcing has been recently studied in [18]. It was shown that global regularity of the velocity field is ensured if and only if

$$\Gamma_0(x) - 2\Omega\omega_0(x) < \Omega^2, \quad \forall x \in \mathbb{R}^2,$$

where $\omega_0 = \nabla_x \times U_0$ is the initial vorticity, and $\Gamma_0 := (\lambda_2(0) - \lambda_1(0))^2$ with initial eigenvalues $\lambda_i(0) := \lambda_i(\nabla_x U_0)$ reflects the initial spectral gap.

It is known that in a rotational reference frame, in addition to the Coriolis forcing $-\Omega JU$, the 'particle' also experiences the centrifugal force, which depends solely on the rotation rate Ω and the distance of the particle to the rotation axis, ΩJx. However, for flows on rotating planet, the centrifugal force is unimportant since it is counter-balanced by the gravitational force pointing toward the planet's center. Therefore the result obtained in [18] applies to, among others, fluid dynamic problems on rotating planet.

In other contexts, say the above mentioned BECs, the centrifugal force has to be taken into consideration, as described by (1.1)–(1.2). The main quest in this paper is whether the action of dispersive rotational forcing together with the underlying potential prevents the generic finite time breakdown of nonlinear convection. Note that the system (1.1)–(1.2) is weakly coupled since the velocity can be solved independently from the density equation, therefore we may discuss velocity regularity for (1.1) independent of (1.2). We show that (1.1) admits global smooth solutions for and only for a subset of generic initial configurations, U_0. Thus, global regularity depends on whether the initial configuration crosses an intrinsic critical threshold, which is quantified in terms of the initial spectral gap, $\Gamma_0 := (\lambda_2(0) - \lambda_1(0))^2$, as well as the initial divergence $\mathrm{div}_x(U_0) = \lambda_1(0) + \lambda_2(0)$.

Theorem 1.1 (Critical Thresholds). *Consider the 2D rotational model* (1.1) *with* $\Omega > 0$. *Then the solution of* (1.1) *with initial velocity* $U(x, 0) = U_0(x)$ *remains smooth for all time if and only if the initial velocity gradient* $\nabla_x U_0$ *satisfies*

- *for isotropic trapping potential* $V(x) = \frac{1}{2}\omega^2|x|^2$

$$\Gamma_0(x) < 0, \quad \forall x \in \mathsf{IR}^2. \tag{1.4}$$

- *for isotropic repulsive potential* $V(x) = -\frac{1}{2}\omega^2|x|^2, \quad \omega \geq 0$

$$\Gamma_0(x) < 0 \quad \text{or} \quad \text{div}_x(U_0) \geq \sqrt{\Gamma_0(x)} - 2\omega, \forall x \in \mathsf{IR}^2. \tag{1.5}$$

Several remarks are in order.

1. We note that in the semiclassical regime of the Schrödinger equation the velocity gradient is the Hessian of the phase S and has only real eigenvalues, i.e., $\Gamma_0 \geq 0$. Therefore Theorem 1.1 shows that the $O(\epsilon^2)$ perturbations ignored in the formal derivation of the WKB system from (1.3) will become significant in finite time. Such results, however, do not bear directly on some current BEC models since in the physical setting as stated in [1], the particles are confined within a bounded domain. It would be of interest to refine the estimate in this paper, by taking into account the above confinements.

2. The critical threshold (1.4) or (1.5) is independent of the initial vorticity $\omega_0 := \nabla_x \times U_0$ for the case of both trapping and repulsive potential, which is in contrast to the critical thresholds presented in [18].

3. The equation (1.1) with repulsive potential does admit global smooth solutions with negative initial divergence in contrast to the free transport ($\Omega = 0, \omega = 0$) equation. It follows that rotation with time-independent potential prevents finite time breakdown by a large initial rotation ($\Gamma_0 < 0$) or the divergence which is not too negative.

4. If we set $\Omega = 0$, i.e., (1.1) with null rotation, then the equation (1.1) with trapped potential is reduced to the system

$$\partial_t U + U \cdot \nabla_x U = -\omega^2 x,$$

with critical threshold $\Gamma_0 < 0$. To interpret results stated in Theorem 1.1 in this simple setting, we observe that the velocity gradient $M := \nabla_x U$ satisfies

$$\partial_t M + (U \cdot \nabla_x)M + M^2 = -\omega^2 I_{2\times 2}.$$

Note that the particle path is defined by

$$\begin{cases} \frac{dX}{dt} = U(X,t), \quad X(0) = \alpha, \\ \frac{dU}{dt} = -\omega^2 X, \quad U(0) = U_0(\alpha), \end{cases}$$

which can be explicitly solved as

$$X = \alpha \cos(\omega t) + \frac{U_0}{\omega}\sin(\omega t), \quad U(t) = U_0(\alpha)\cos(\omega t) - \omega\alpha\sin(\omega t).$$

Along this particle path eigenvalues of M evolve according to

$$\frac{d\lambda}{dt} + \lambda^2 = -\omega^2, \quad \lambda = \lambda_i, \quad i = 1, 2,$$

which has global bounded solution if and only if the initial data $\lambda_0(\alpha)$ is purely complex, that is $\Gamma_0(\alpha) < 0, \quad \forall \alpha \in \mathrm{IR}^2$. For the repulsive potential the velocity field evolves as

$$\partial_t U + U \cdot \nabla_x U = \omega^2 x,$$

with critical threshold $\{\Gamma_0 < 0\} \cup \{\mathrm{dix}_x(U_0) \geq \sqrt{\Gamma_0} - 2\omega\}$. Note that the particle path defined by

$$\begin{cases} \frac{dX}{dt} = U(X, t), \quad X(0) = \alpha, \\ \frac{dU}{dt} = \omega^2 X, \quad U(0) = U_0(\alpha), \end{cases}$$

has an explicit expression

$$X = \alpha \cosh(\omega t) + \frac{U_0}{\omega} \sinh(\omega t), \quad U(t) = U_0(\alpha) \cosh(\omega t) - \omega \alpha \sinh(\omega t).$$

The velocity gradient $M := \nabla_x U$ satisfies

$$\partial_t M + (U \cdot \nabla_x) M + M^2 = \omega^2 I_{2\times 2}.$$

Along the particle path the eigenvalue of M evolves as

$$\frac{d\lambda}{dt} + \lambda^2 = \omega^2, \quad \lambda = \lambda_i, \quad i = 1, 2.$$

which has global bounded solution iff the initial data $\lambda_0(\alpha)$ is purely complex or $\lambda_0(\alpha) \geq -\omega$ which is equivalent to $\{\Gamma_0 < 0\} \cup \{\Gamma_0 \geq 0, \quad \mathrm{div}_x(U_0) \geq \sqrt{\Gamma_0} - 2\omega\}$.

To put our study in a proper perspective we recall that there has been a considerable amount of literature available on the global behavior of the nonlinear convection driven by rotational forcing and related problems, from rotating shallow-water models [10, 14, 21] to rotating incompressible Euler and Navier-Stokes equations [2, 3, 8, 5]. The common feature of the flows studied in this context are rotation dominated flow for which the Rossby number Ω^{-1} is small. It is well known that large-scale atmospheric (or oceanic) fields are in permanent process of Rossby (or geostrophic) adjustment [19]. The flow structure has been extensively studied in terms of Ω^{-1}, say in [2, 10], based on the averaging of the interaction of the fast waves of the rotating Euler equation, two dimensional structures were shown to emerge in the limit $\Omega^{-1} \to 0$; for bounds of the vertical gradients of the Lagrangian displacement that vanish linearly with the maximal local Rossby number [8]; for a nonlinear theory of geostrophic adjustment for the rotating shallow-water model under the assumption of the smallness of the Rossby number [21]; consult [14] for the analysis of an approximation of the rotating shallow-water equation.

It is known that the classical stability analysis is not enough to reveal the conditional stability phenomena in nonlinear problems. To address such difficulty we advocated, in [9, 15, 16], a new notion of critical threshold (CT) which describes conditional stability, where the answer to the question of global vs. local existence depends on whether the initial configuration crosses an intrinsic critical threshold. The critical threshold (CT) was completely characterized for the 1D Euler-Poisson system in terms of the relative size of the initial velocity slope and the initial density; consult [15, 27] for the CT in convolution models for conservation laws. Moving to the multi-D setup, one has first to identify the proper quantities which govern the critical threshold phenomena. In [17, 18] we have shown that these quantities depend in an essential manner on the *eigenvalues* of the velocity gradient matrix, $\lambda(\nabla_x U)$.

The critical threshold for the current rotation model can also be obtained, in a straightforward manner, through a Lagrangian flow formulation. This is summarized in the Theorem 1.2 below. In Section 3 we prove

Theorem 1.2 (Lagrangian Dynamics of Velocity Field). *Let the deformed flow map and associated velocity field be determined by*

$$\begin{aligned} \dot{X}_\alpha &:= \frac{dX_\alpha}{dt} = U(X_\alpha) - \Omega J X_\alpha, \\ \dot{U}(X_\alpha) &= -\Omega J U(X_\alpha) - \nabla_x V \end{aligned}$$

with initial position $X_\alpha(0) = \alpha$ and initial velocity $U(\alpha, 0) = U_0(\alpha)$. Then,

- *for trapping potential, $V = \frac{\omega^2}{2}|x|^2$*

$$\begin{aligned} X_\alpha(t) &= e^{-\Omega J t}\left(\cos(\omega t)\alpha + \frac{\sin(\omega t)}{\omega} U_0(\alpha)\right), \\ U(t) &= e^{-\Omega J t}\left(-\omega \sin(\omega t)\alpha + \cos(\omega t) U_0(\alpha)\right), \end{aligned}$$

  which are periodic with periodicity of least common of $2\pi/\Omega$ and $2\pi/\omega$;

- *for repulsive potential, $V = -\frac{\omega^2}{2}|x|^2$*

$$\begin{aligned} X_\alpha(t) &= e^{-\Omega J t}\left(cosh(\omega t)\alpha + \frac{sinh(\omega t)}{\omega} U_0(\alpha)\right), \\ U(t) &= e^{-\Omega J t}\left(-\omega sinh(\omega t)\alpha + cosh(\omega t) U_0(\alpha)\right), \end{aligned}$$

- *for null potential, $V = 0$*

$$\begin{aligned} X_\alpha(t) &= e^{-\Omega J t}\left(\alpha + t U_0(\alpha)\right), \\ U(t) &= e^{-\Omega J t} U_0(\alpha). \end{aligned}$$

For sub-critical initial data, i.e., (1.4) *or* (1.5) *is satisfied, the above deformed flow maps are invertible and the associated velocity fields remain regular for all time.*

Finally we conclude in Section 5 with the density behavior along the deformed flow map.

Theorem 1.3. *Let U be the velocity field governed by* (1.1) *and ρ be the associated density satisfying* (1.2). *The density remains bounded if and only if the velocity* *field remains smooth, which is ensured by the sub-critical initial data, i.e.,* (1.4) *or* (1.5)*. Moreover the density can be explicitly expressed along the deformed flow map by*

$$\rho(X_\alpha(t), t) = \frac{\rho_0(\alpha)}{I(t)},$$

where

- *for trapped potential* $V(x) = \frac{1}{2}\omega^2|x|^2$*,*

$$I(t) := \cos^2(\omega t) + \frac{\nabla_x \cdot U_0}{\omega}\cos(\omega t)\sin(\omega t) + \frac{det(\nabla_x U_0)}{\omega^2}\sin^2(\omega t),$$

- *for repulsive potential* $V(x) = -\frac{1}{2}\omega^2|x|^2$*,*

$$I(t) := cosh^2(\omega t) + \frac{\nabla_x \cdot U_0}{\omega} cosh(\omega t) sinh(\omega t) + \frac{det(\nabla_x U_0)}{\omega^2} sinh^2(\omega t),$$

- *for null potential* $V(x) = 0$*,*

$$I(t) := 1 + \nabla_x \cdot U_0 t + det(\nabla_x U_0)t^2.$$

After the present section, Section 2 is devoted to a formal passage between the rotational Schrödinger equation and the hydrodynamic system (1.1)–(1.2), identified as the semi-classical limiting system of the underlying Schrödinger equation. In Section 3 and 4 below, we quantify critical thresholds using both the Lagrangian and Eulerian formulations.

2. WKB system of rotational Schrödinger equation

If a harmonic trap potential is considered, the rotational Gross–Pitaevskii equation becomes

$$\begin{aligned} &i\hbar\partial_t\psi(x,t) \\ &\quad = -\frac{\hbar^2}{2m}\Delta\psi(x,t) + i\hbar\Omega\langle x^\perp, \nabla_x\psi\rangle + \frac{m}{2}|\omega\cdot x|^2\psi(x,t) + NP_0|\psi(x,t)^2|\psi(x,t), \end{aligned}$$

where $x \in \mathbb{R}^2$ is the spatial coordinate vector, m is the atomic mass, $\hbar$ is the Planck constant, N is the number of atoms in the condensate, and ω denotes the trap frequency. P_0 describes the interaction between atoms in the condensate and has the form

$$P_0 = \frac{4\pi\hbar^2 a}{m}$$

where a is the s-wave scattering length. Let L be the characteristic "length" of the condensate. Under the following scaling

$$(x, t, \psi) \to (L^{-1}x, t, L^{3/2}\psi(x,t))$$

the GPE is reduced to

$$\begin{aligned} &i\epsilon\partial_t\psi(x,t) \\ &\quad = -\frac{\epsilon^2}{2}\Delta\psi(x,t) + i\epsilon\Omega\langle x^\perp, \nabla_x\psi\rangle + \frac{1}{2}|\omega\cdot x|^2\psi(x,t) + \delta\epsilon^{5/2}|\psi(x,t)^2|\psi(x,t), \end{aligned}$$

where $\epsilon = \frac{\hbar}{mL^2}$ and $\delta = 4\pi aN\sqrt{\frac{m}{\hbar}}$.

We are interested in the high frequency propagating waves for the linearized rotational Schödinger equations with fast space-time scales

$$i\epsilon\partial_t\psi^\epsilon = -\frac{\epsilon^2}{2}\Delta\psi^\epsilon + i\epsilon\Omega\langle Jx, \nabla_x\psi\rangle + V(x)\psi^\epsilon, \quad x\in \mathrm{IR}^2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.1}$$

subject to the high frequency initial data

$$\psi^\epsilon(x,0) = A_0(x)\exp\left(i\frac{S_0(x)}{\epsilon}\right), \tag{2.2}$$

where ψ^ϵ is the complex wave field, $V(x)$ is a given potential function, and $\epsilon > 0$, appearing in both the equation and the initial data, denotes a re-scaled Planck constant.

The connection between Schrödinger equations and the classical hydrodynamics was already noted in 1927 by Madelung, in the context of semi-classical limit of the nonlinear Schrödinger equation. To this end, one identifies two physically relevant observable quantities–the fluid density $\rho := |\psi|^2$, and the fluid velocity $U = \epsilon\nabla_x \arg(\psi)$. Indeed, introducing S as the phase of the wave function ψ^ϵ, the WKB solution is sought in the form of

$$\psi^\epsilon(x,t) = \sqrt{\rho(x,t)}\exp(iS(x,t)/\epsilon).$$

Insertion of this ansatz into (2.1) and balance terms of $O(1)$ order in ϵ with separate real and imaginary parts gives the separate equations for ρ and S. The phase function S will satisfy a perturbed nonlinear first order equation of the Hamilton-Jacobi type

$$\partial_t S + \frac{1}{2}|\nabla_x S|^2 - \Omega\langle Jx, \nabla_x S\rangle + V(x) = \frac{\epsilon^2}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}, \tag{2.3}$$

and the position density ρ solves a forced transport equation

$$\partial_t\rho + \nabla\cdot(\rho\nabla S) = \Omega\langle Jx, \nabla_x\rho\rangle.$$

The resulting phase equation (2.3) amounts to a dispersive regularization with the square of ϵ on the right of (2.3) playing the role of the amplitude of dispersion. If we argue formally that this $O(\epsilon^2)$ term in the phase equation is negligible as $\epsilon\to 0$, then the corresponding limiting system when rewritten for density ρ and velocity $U = \nabla_x S$ is nothing but the system (1.1)–(1.2). The dispersive regularization term in the phase equation represents a quantum correction, and the system (1.1)–(1.2) governing the observables (ρ, U) describes the quantum Eulerian dynamics in rotating frame.

The above argument is, of course, only formal. The Madelung's transformation relies on the assumption that the amplitude ρ does not vanish and that the phase S remains nonsingular, for otherwise the transformation is not well-defined and the hydrodynamic system becomes singular, even though the Schrödinger equation itself is still regular. One of interesting issues here is to justify such a dispersive limit. The asymptotics of the observable quantities as $\epsilon \to 0$ is known as 'semi-classical', expressing the passage from quantum to Newton mechanics, where the time and distance scales become large enough relative to Planck's constant.

As indicated by the nonlinear convection term the rotational momentum equation (1.1) in general develops a singularity in finite time. The justification of the 'semi-classical' limit hinges on whether we are able to establish the global regularity for the limiting system (1.1)–(1.2), which is actually the main goal of this work.

3. Lagrangian dynamics of velocity field

This section is devoted to the study of structures of the deformed flow map as well as the associated velocity field. We focus on the isotropic quadratic potential, i.e., $\omega_1^2 = \omega_2^2$ and distinguish cases of trapped, repulsive as well as the null potential.

Let $x = X_\alpha(t)$ be the deformed flow map due to the centrifugal force with initial position $X_\alpha(0) = \alpha$, and $U(t) := U(X_\alpha(t), t)$ be the associated velocity field, then it follows from (1.1) that

$$\dot{X}_\alpha := \frac{dX_\alpha}{dt} = U(t) - \Omega J X_\alpha, \quad X_\alpha(0) = \alpha, \tag{3.1}$$

$$\dot{U} := \frac{dU(X_\alpha, t)}{dt} = -\Omega J U - \nabla_x V(X_\alpha), \quad U(0) = U_0(\alpha). \tag{3.2}$$

3.1. Trapping potential

For isotropic trapping potential $V(x) = \frac{1}{2}\omega^2|x|^2$, the velocity U in (3.2) evolves as

$$\dot{U} = -\Omega J U - \omega^2 X_\alpha.$$

Let $Y(t) := e^{\Omega J t} X_\alpha$ and $V(t) := e^{\Omega J t} U$, a simple calculation yields

$$\dot{Y} = e^{\Omega J t}(\dot{X}_\alpha + \Omega J X_\alpha) = e^{\Omega J t} U = V,$$
$$\dot{V} = e^{\Omega J t}(\dot{U} + \Omega J U) = -\omega^2 e^{\Omega J t} X_\alpha = -\omega^2 Y.$$

For the space-dependent component Y one has

$$\ddot{Y} + \omega^2 Y = 0,$$

subject to the initial condition

$$Y(t=0) = X_\alpha(t=0) = \alpha, \quad V(t=0) = U_0(\alpha).$$

Its explicit solution is

$$Y = \cos(\omega t)\alpha + \sin(\omega t)U_0(\alpha)/\omega, \quad V = \dot{Y} = -\omega \sin(\omega t)\alpha + \cos(\omega t)U_0(\alpha),$$

where $U_0(\alpha)$ is the initial velocity at location α. This together with the used transformation $(X_\alpha, U) \to (Y, V)$ leads to the deformed flow map

$$X_\alpha(t) = e^{-\Omega J t} Y = e^{-\Omega J t}\left(\cos(\omega t)\alpha + \frac{\sin(\omega t)}{\omega} U_0(\alpha)\right),$$

and the velocity field

$$U(t) = e^{-\Omega J t} V = e^{-\Omega J t}\left(-\omega \sin(\omega t)\alpha + \cos(\omega t) U_0(\alpha)\right).$$

Note that the exponential function

$$e^{-\Omega J t} = \begin{pmatrix} \cos(\Omega t) & -\sin(\Omega t) \\ \sin(\Omega t) & \cos(\Omega t) \end{pmatrix},$$

is periodic with periodicity $2\pi/\Omega$. Therefore the deformed flow map as well as the associated velocity field, as a product of two periodic functions with periodicity $2\pi/\Omega$ and $2\pi/\omega$, respectively, is periodic if and only if they remain smooth and the ratio, ω/Ω, is a rational number. In fact the deformed flow map determines the unique smooth velocity field U if and only if the indicator matrix,

$$\nabla_\alpha X_\alpha(t) = e^{-\Omega J t}\left(\cos(\omega t) I_{2\times 2} + \frac{\sin(\omega t)}{\omega} \nabla_x U_0(\alpha)\right),$$

remains nonsingular. A straightforward calculation, in virtue of $\det(e^{-\Omega J t}) = 1$, gives its determinant as

$$\begin{aligned} \det\left(\nabla_\alpha X_\alpha(t)\right) &= \det(e^{-\Omega J t}) \cdot \det\left(\cos(\omega t) I_{2\times 2} + \frac{\sin(\omega t)}{\omega} \nabla_x U_0(\alpha)\right) \\ &= \cos^2(\omega t) + \frac{\nabla_x \cdot U_0}{\omega} \cos(\omega t)\sin(\omega t) + \frac{\det(\nabla_x U_0)}{\omega^2} \sin^2(\omega t). \quad (3.3) \end{aligned}$$

Thus $\nabla_\alpha X_\alpha(t)$ remains nonsingular for all time if and only if $\det\left(\nabla_\alpha X_\alpha(t)\right) \neq 0$, i.e.,

$$(\nabla_x U_0)^2 - 4\det(\nabla U_0) < 0. \quad (3.4)$$

The spectral gap $\Gamma_0 = (\lambda_{02} - \lambda_{01})^2$ relates the determinant and the divergence as

$$\Gamma_0 = (\nabla_x \cdot U_0)^2 - 4\det(\nabla_\alpha U_0).$$

This when applied to the above inequality (3.4) implies that the initial spectral gap is imaginary, i.e.,

$$\Gamma_0(\alpha) < 0, \quad \forall \alpha \in \mathbb{R}^2,$$

which is exactly the critical threshold (1.4) stated in Theorem 1.1.

3.2. Repulsive potential

For the isotropic repulsive potential $V(x) = -\frac{1}{2}\omega^2|x|^2$ the deformed flow map and the velocity field satisfy a closed ODE system

$$\dot{X}_\alpha = U(t) - \Omega J X_\alpha,$$
$$\dot{U} = -\Omega J U + \omega^2 X_\alpha,$$

with initial position $X_\alpha(0) = \alpha$ and the initial velocity $U_0(\alpha)$. By the transformation approach as performed previously we can obtain for $Y(t) := e^{\Omega J t} X_\alpha$ and $V(t) := e^{\Omega J t} U$

$$\dot{Y} = V, \quad \dot{V} = \omega^2 Y.$$

For the component Y one has

$$\ddot{Y} - \omega^2 Y = 0,$$

its general solution is

$$Y = Ae^{-\omega t} + Be^{\omega t}, \quad V = \dot{Y} = -A\omega e^{-\omega t} + B\omega e^{\omega t}.$$

Using the initial condition

$$Y(t=0) = \alpha, \quad V(t=0) = U_0(\alpha),$$

one gets

$$A = \frac{1}{2}(\alpha + U_0(\alpha)/\omega), \qquad B = \frac{1}{2}(\alpha - U_0(\alpha)/\omega).$$

Therefore we obtain the deformed flow map

$$X_\alpha(t) = e^{-\Omega J t}\left(\cosh(\omega t)\alpha + \frac{\sinh(\omega t)}{\omega} U_0(\alpha)\right),$$

and the velocity field

$$U(t) = e^{-\Omega J t}\left(-\omega \sinh(\omega t)\alpha + \cosh(\omega t) U_0(\alpha)\right).$$

The deformed flow map determines the unique smooth velocity field U if and only if the indicator matrix,

$$\nabla_\alpha X_\alpha(t) = e^{-\Omega J t}\left(\cosh(\omega t) I_{2\times 2} + \frac{\sinh(\omega)t}{\omega} \nabla_x U_0(\alpha)\right)$$

remains nonsingular. A straightforward calculation, in virtue of $\det(e^{-\Omega J t}) = 1$, gives its determinant as

$$\begin{aligned}
\det\left(\nabla_\alpha X_\alpha(t)\right) &= \det\left(e^{-\Omega J t}\right) \cdot \det\left(\cosh(\omega t) I_{2\times 2} + \frac{\sinh(\omega)t}{\omega} \nabla_x U_0(\alpha)\right) \\
&= \cosh^2(\omega t) + \frac{\nabla_x \cdot U_0}{\omega} \cosh(\omega t)\sinh(\omega t) + \frac{\det(\nabla_x U_0)}{\omega^2} \sinh^2(\omega t) \qquad (3.5) \\
&= \cosh^2(\omega t) \cdot F(\tanh(\omega t)),
\end{aligned}$$

where

$$F(\tau) = 1 + \frac{\nabla_x \cdot U_0}{\omega}\tau + \frac{\det(\nabla_x U_0)}{\omega^2}\tau^2.$$

Note that $\tanh(\omega t)$ runs over $[0,1]$ for $t \in [0,\infty)$, also the possible zero of F is attained at

$$\tau^* = \omega \cdot \frac{-\nabla_x \cdot U_0 \pm \sqrt{(\nabla_x \cdot U_0)^2 - 4\det(\nabla_x U_0)}}{2\det(\nabla_x U_0)} = \frac{-2\omega}{d_0 \pm \sqrt{\Gamma_0}}, \quad d_0 := \mathrm{div}_x(U_0).$$

These facts show that $\det(\nabla_\alpha X_\alpha(t)) \neq 0$ for all time if and only if either $\Gamma_0 < 0$ (τ^* is complex) or $d_0 \geq \sqrt{\Gamma_0} - 2\omega$ for $\Gamma_0 \geq 0$ ($\tau^* \notin [0,1]$), which is the critical threshold (1.5) stated in Theorem 1.1.

In order to have a complete picture we discuss the case with zero potential $\omega = 0$. In this case we have the deformed flow map

$$X_\alpha(t) = e^{-\Omega J t}[\alpha + tU_0(\alpha)]$$

and the velocity field

$$U(t) = e^{-\Omega J t}U_0(\alpha).$$

Therefore the indicator matrix is

$$\nabla_\alpha X_\alpha(t) = e^{-\Omega J t}\left(I_{2\times 2} + t\nabla_x U_0(\alpha)\right).$$

The solution remains bounded if and only if $\nabla_\alpha X_\alpha(t)$ remains nonsingular, i.e.,

$$\det\left(\nabla_\alpha X_\alpha(t)\right) = 1 + t\nabla_x \cdot U_0(\alpha) + t^2 \det(\nabla_x U_0) \neq 0 \tag{3.6}$$

for all time $t > 0$. The possible zero of the above function is

$$t^* = \frac{-\nabla_x \cdot U_0 \pm \sqrt{(\nabla_x \cdot U_0)^2 - 4\det(\nabla_x U_0)}}{2\det(\nabla_x U_0)} = \frac{-2}{d_0 \pm \sqrt{\Gamma_0}}.$$

Therefore $\det\left(\nabla_\alpha X_\alpha(t)\right) \neq 0$ iff either $\Gamma_0 < 0$ (complex t^*) or $d_0 \geq \sqrt{\Gamma_0}$ for $\Gamma_0 \geq 0$ ($t^* < 0$). Note also that in this case the deformed particle path is no longer a periodic function, and the velocity field is still periodic with periodicity $2\pi/\Omega$.

4. Velocity gradient

Along the smooth deformed particle path we now track the dynamics of the velocity gradient as well as the associated density function. Rewrite the system (1.1)–(1.2) as

$$\begin{aligned}(\partial_t + (U - \Omega Jx)\cdot\nabla_x)\rho &= -\rho\nabla\cdot U,\\ (\partial_t + (U - \Omega Jx)\cdot\nabla_x)U &= -\Omega JU - \nabla_x V.\end{aligned}$$

Let velocity gradient be $M := \nabla_x U$, take the gradient of the momentum equation one has

$$\left(\partial_t + (U - \Omega Jx)\cdot\nabla_x\right)M + M^2 = -\nabla \otimes \nabla V(x). \tag{4.1}$$

Consider a generalized nonlinear transport equation, $\partial_t U + (U - \Omega Jx) \cdot \nabla_x U = F$, and we trace the evolution of $\nabla_x U$ in terms of its eigenvalues, $\lambda := \lambda(\nabla_x U)(t, x)$. The following result is a generalized version of the Spectral Dynamics Lemma 3.1 stated in [16].

Lemma 4.1. *Let $\lambda := \lambda(\nabla_x U)(t, x)$ denote an eigenvalue of $\nabla_x U$ with corresponding left and right normalized eigenpair, $\langle \ell, r\rangle = 1$. Then λ is governed by the forced Riccati equation*

$$\partial_t \lambda + (U - \Omega Jx) \cdot \nabla_x \lambda + \lambda^2 = \langle \ell, \nabla_x F r\rangle.$$

We shall use Lemma 4.1 to obtain the remarkable explicit formulae for the critical threshold surfaces summarized in the main Theorems 1.1.

Applying the spectral dynamics Lemma 4.1 to (4.1), we obtain the spectral dynamic equations

$$\partial_t \lambda_1 + (U - \Omega Jx) \cdot \nabla_x \lambda_1 + \lambda_1^2 = -\langle l_1, \nabla \otimes \nabla V(x) r_1\rangle, \tag{4.2}$$

$$\partial_t \lambda_2 + (U - \Omega Jx) \cdot \nabla_x \lambda_2 + \lambda_2^2 = -\langle l_2, \nabla \otimes \nabla V(x) r_2\rangle, \tag{4.3}$$

where $\lambda_i, i = 1, 2$ are eigenvalues of the velocity gradient field $\nabla_x U$ associated with left (row) eigenvectors l_i and right (column) eigenvectors r_i.

Note that in the current setting, the potential is quadratic and isotropic and therefore

$$\nabla_x \otimes \nabla_x V(x) = \pm\omega^2 I_{2\times 2}.$$

Equipped with the above relations we end up with a closed decoupled system for (λ_1, λ_2) along the deformed particle path $X(\alpha, t)$ (here and below $\dot{} \equiv \partial_t + (U - \Omega JX) \cdot \nabla_x)$

$$\dot\lambda + \lambda^2 = \pm\omega^2, \quad \lambda = \lambda_1(t) \quad \text{or} \quad \lambda_2(t). \tag{4.4}$$

Here the eigenvalue $\lambda(t)$ could be a complex function in time. One may introduce the real quantities $(\Gamma, d) = ((\lambda_2 - \lambda_1)^2, \lambda_1 + \lambda_2)$, which solve a closed coupled system

$$d' = -\frac{d^2 + \Gamma}{2} \pm 2\omega^2, \tag{4.5}$$

$$\Gamma' = -2d\Gamma. \tag{4.6}$$

The question here is what conditions should be imposed on the initial data (d_0, Γ_0) so that (d, Γ) are bounded globally in time. This being said we still use eigenvalues to identify such critical condition since equations for λ_1 and λ_2 are decoupled, even though they might be complex functions.

There are three cases to be considered:

(1) Trapping potential, $\dot\lambda + \lambda^2 = -\omega^2$.
The Ricatti type equation above has global bounded solution only if the initial data are complex, which is equivalent to the negativity of the spectral gap indicator,

i.e.,

$$\Gamma_0(\alpha) < 0, \quad \alpha \in \mathrm{IR}^2.$$

Also formally one could find the time dependent eigenvalues

$$\lambda(t) = \omega \tan(\tan^{-1}(\lambda(0)/\omega) - \omega t), \quad \lambda = \lambda_i \quad i = 1, 2. \tag{4.7}$$

The eigenvalues are periodic functions in time with periodicity π/ω. Thus the divergence $d = \lambda_1(t) + \lambda_2(t)$ is also periodic and bounded. It follows that the boundedness of the divergence implies the boundedness of the whole velocity gradient: from (4.1) we see that the anti-trace of M, $r = v_x + u_y$, the anti-vorticity $u_x - v_y$ as well as the vorticity, $u_y - v_x$, satisfy the same transport equation $\dot{r} + dr = 0$, these together with bounded divergence $d := \nabla_x \cdot U$ yield the boundedness of velocity gradient $\nabla_x U$. The whole velocity gradient is also periodic in time with periodicity $2\pi/\omega$. Also the boundedness of the divergence along the particle path enables us to conclude that the density is also bounded.

(2) Null potential, $\dot{\lambda} + \lambda^2 = 0$.
The eigenvalues remain bounded for all time if and only if either the initial eigenvalue is complex or $\lambda_0 \geq 0$ with bounded solution given by

$$\lambda(t) = \frac{\lambda_0}{1 + \lambda_0 t}, \quad \lambda = \lambda_i, i = 1, 2. \tag{4.8}$$

These conditions when expressed as the divergence $d_0 = \mathrm{div}_x(U_0)$ and the spectral gap Γ_0 are equivalent to the critical thresholds described in (1.5), i.e.,

$$\Gamma_0(x) < 0 \quad \text{or} \quad d_0(x) \geq \sqrt{\Gamma_0(x)}, \quad \forall x \in \mathrm{IR}^2.$$

(3) Repulsive potential
In this case the eigenvalue $\lambda(t)$ satisfy $\dot{d} = \omega^2 - \lambda^2$, which has global bounded solution for all time if and only if either the initial eigenvalues are complex or real bigger than $-w$. This condition on eigenvalues when expressed in terms of the divergence and the spectral gap is equivalent to (1.5), i.e.,

$$\Gamma_0(\alpha) < 0 \quad \text{or} \quad d_0(\alpha) \geq \sqrt{\Gamma_0(\alpha)} - 2\omega, \quad \forall \alpha \in \mathrm{IR}^2.$$

In this setting the bounded eigenvalues are determined by

$$\lambda(t) = \frac{\lambda_0(\alpha)\cosh(\omega t) + \omega \sinh(\omega t)}{\lambda_0(\alpha)\sinh(\omega t) + \omega \cosh(\omega t)}\omega, \quad \lambda = \lambda_i, \quad i = 1, 2. \tag{4.9}$$

As discussed for trapping potential case the whole velocity gradient is of dependent on the divergence, therefore on eigenvalues.

5. Position density

Lemma 5.1. *Let $A(t)$ be a smooth, nonsingular 2×2 matrix-valued function, then*

$$Trace\left(\frac{dA}{dt}A^{-1}\right) \equiv \frac{d}{dt}\left(ln(\det(A))\right).$$

This relation can be justified by a straightforward calculation. Using this fact we establish the following

Lemma 5.2. *For the sub-critical initial velocity gradient, i.e., one of* (1.4)–(1.5) *is satisfied, the divergence of the velocity remains bounded. Moreover*

$$\exp\left(\int_0^t \nabla_x \cdot U(\tau)d\tau\right) = \det\left(\nabla_\alpha X_\alpha(t)\right). \tag{5.1}$$

Proof. It follows from

$$\dot{X_\alpha} = U(X_\alpha, t) - \Omega J X_\alpha$$

that

$$\frac{d}{dt}\left(\nabla_\alpha X_\alpha(t)\right) = \left(\nabla_x U - \Omega J\right) \cdot \nabla_\alpha X_\alpha(t).$$

Since $\nabla_\alpha X_\alpha(t)$ remains nonsingular for sub-critical initial velocity gradient, one may rewrite the above as

$$\nabla_x U - \Omega J = \frac{d}{dt}\left(\nabla_\alpha X_\alpha(t)\right) \cdot \left(\nabla_\alpha X_\alpha(t)\right)^{-1}.$$

Taking the trace on both sides and using Lemma 5.1 we obtain

$$\nabla_x \cdot U = \frac{d}{dt}\left(ln(\det(\nabla_\alpha X_\alpha(t)))\right).$$

Note that at initial time $\det(\nabla_\alpha X_\alpha(t)) = 1$. Hence the asserted relation (5.1) follows from the integration of this equation over $[0, t]$. □

Equipped with the relation (5.1) we are in a position to study the behavior of the position density. Along the deformed flow map $X_\alpha(t)$ the density is known to satisfy

$$\frac{d\rho}{dt} = -\rho \nabla_x \cdot U.$$

Upon integration one gets

$$\rho(X_\alpha(t), t) = \rho_0(\alpha) \cdot \exp\left(-\int_0^t \nabla_x \cdot U(\tau)d\tau\right),$$

which when combined with Lemma 5.2 and the explicit relations, (3.3), (3.5) and (3.6), for $\det(\nabla_\alpha X_\alpha(t))$ found in Section 2 establishes the results asserted in Theorem 1.3.

As an alternative one may use the relation $\nabla_x \cdot U = \lambda_1 + \lambda_2$ to get

$$\rho(t) = \rho_0 \exp\left(-\int_0^t (\lambda_1(\tau) + \lambda_2(\tau))d\tau\right),$$

which, based on the eigenvalues given in (4.7)–(4.9), again leads to the asserted formula for position density stated in Theorem 1.3.

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Hailiang Liu
Department of Mathematics
Iowa State University
Ames, IA 50011
USA
e-mail: hliu@iastate.edu

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